Objective A4: Student can identify features of a function given as a table or a graph, including the domain and range, the intercepts, the intervals where the function is increasing/decreasing, and the intervals where the function is positive/negative.

Key Vocabulary Words

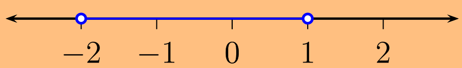
- Domain
- Range
- Horizontal intercept (zero, root, x-intercept)
- Vertical intercept (y-intercept)

We have already discussed how to find function values given a numerical, graphical, symbolic, or verbal representation, but what if we wanted to be able to more broadly describe the behavior of the function?

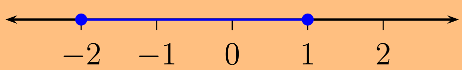
In order to do this, let's first recall some interval notation:

Interval Notation

- $(-2, 1)$ means all numbers greater than -2 and less than 1, which looks like $-2 < x < 1$ or



- $[-2, 1]$ means all numbers greater than or equal to -2 and less than or equal 1, which looks like $-2 \leq x \leq 1$ or



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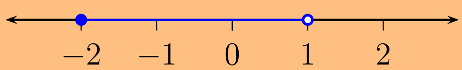

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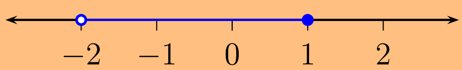


Note that whenever we have a bracket, this means that the endpoint *is* included (denoted by a filled-in circle) and a parenthesis means that the endpoint is *not* included (denoted by an open circle).

For all intervals, the smaller value always comes first, e.g. we write $(-2, 1)$ and not $(1, -2)$. When we have an interval that includes ∞ or $-\infty$, we always use parentheses around the ∞ or $-\infty$ when denoting that interval, e.g. $(-\infty, 2]$, $(-4, \infty)$, $(-\infty, \infty)$.

Domain and Range

Now we can use interval notation to describe different aspects of graphs for a function $f(x)$. We refer to the set of all allowable inputs (x-values) as the **domain**. The set of all function's output values (y-values) that correspond to the domain is called the **range**.

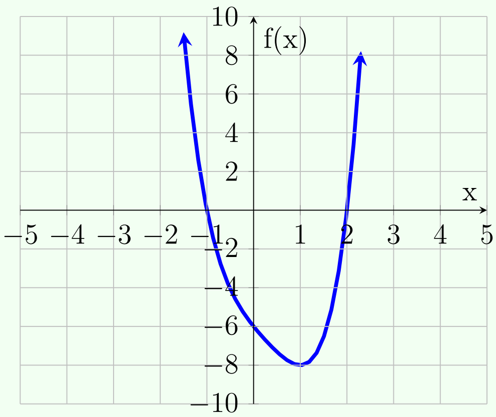
Example 1a: Table
Find the domain and range for the function given by the following table:

x	0	1	2	3	4	5	6	7	8
$p(x)$	21	18	15	12	9	6	3	0	-3

Solution:
All of the inputs x for $p(x)$ are $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, so this set is the domain. The range in this case is the set of all of the outputs, $\{-3, 0, 3, 6, 9, 12, 15, 18, 21\}$.

The braces { } tell us that we are including *only* the values that are listed. For instance, $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ tells us that those nine different values are included whereas $[0, 8]$ would tell us that we also include numbers like 1.5, 2.6, etc.

Example 2a: Graph
Find the domain and range for the function given in the following graph:



Solution:
Since the arrows on the graph tell us that the function extends to all possible inputs x, the domain for this function is $(-\infty, \infty)$. Now we can see that the lowest possible $f(x)$ value occurs at -8 and, like with the x values, the arrows tell us that the outputs continue to ∞, and thus the range is $[-8, \infty)$.

Intercepts

Now within our domain of possible x-values, we want to distinguish the points where the graph of a function crosses the axes. We call the points where $y = 0$ (i.e. where the function crosses the x-axis) the **horizontal intercepts**, *x-intercepts*, *zeros*, or *roots* of the function. We often refer to these points solely by their x-coordinate, since the y is always 0. In contrast, the places where $x = 0$ (i.e. where the function crosses the y-axis) are called **vertical intercepts**, or *y-intercepts*. Similarly, we often refer to these points solely by their y-coordinate, since the x is always 0.

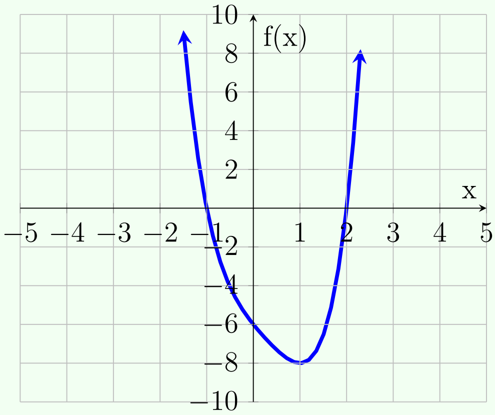
Let's consider a few examples:

Example 1b: Table
Given the following table for the function $p(x)$, find the horizontal and vertical intercepts.

x	0	1	2	3	4	5	6	7	8
$p(x)$	21	18	15	12	9	6	3	0	-3

Solution:
The horizontal intercept occurs when the output $y = 0$, which we can see happens when the input $x = 7$, or $(7, 0)$. The vertical intercept happens when $x = 0$, and so the y-intercept in this case is $y = 21$, or $(0, 21)$.

Example 2b: Graph
Given the following graph for the function $f(x)$, find the horizontal and vertical intercepts.



Solution: The horizontal intercepts (x-intercept) occur when $x = -1, 2$ or the points (-1,0) and (2,0), because those are the places where the graph crosses the x-axis. The vertical intercept (y-intercept) happens when the function crosses the y-axis (so $x = 0$), at $y = -6$ or the point (0,-6).

Positive and Negative

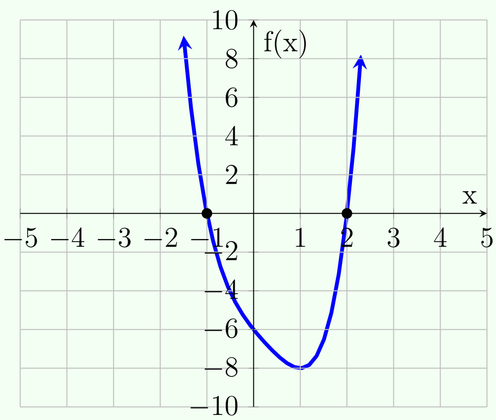
Now that we have found the horizontal intercepts (or zeros) of a function, notice that we can use those to separate the domain into sets where the outputs are positive and sets where the outputs are negative. On a graph, the positive outputs correspond to function values that appear *above* the x-axis, while negative outputs are those that are *below* the x-axis. Outputs that lie *on* the x-axis are *neither* positive *nor* negative since the y-value for those points is 0, which is neither positive nor negative. Note that when we do this process for a graph, we give those sets in terms of the intervals of *input* values where the function values are positive or negative.

Example 1c: Table
Given the following table for the function $p(x)$, find where the function is positive or negative.

x	0	1	2	3	4	5	6	7	8
$p(x)$	21	18	15	12	9	6	3	0	-3

Solution:
Looking at the output values in the table, we can see that $p(x)$ is positive for the inputs $\{0, 1, 2, 3, 4, 5, 6\}$ and negative for the input $\{8\}$.

Example 2c: Graph
Given the following graph for the function $f(x)$, find the intervals where the function is positive or negative.



Solution:
Since we already figured out that the horizontal intercepts are $x = -1, 2$, we can use these to split up the domain $(-\infty, \infty)$. Looking at the graph, we see that *output* values are positive (i.e. the graph is above the x-axis) on the *input* values in the intervals $(-\infty, -1)$ and $(2, \infty)$ and the function is negative (i.e. the graph is below the x-axis) on the interval $(-1, 2)$.

Increasing and Decreasing

Finally, we would like to specify the general behavior of the function, whether it is *increasing* or *decreasing*. Intuitively these refer to what you would think they would mean: increasing means the function values are going up as we go from left to right, and decreasing means the function values are going down as we move from left to right. Technically, a function f is increasing on an interval if the following happens: for any a and b in that interval such that $a < b$, it follows that $f(a) < f(b)$. Similarly a function is decreasing on an interval if for any a and b in that interval such that $a < b$, we have that $f(a) > f(b)$.

Example 1d: Table

Given the following table for the function $p(x)$, find the intervals where the function is increasing or decreasing.

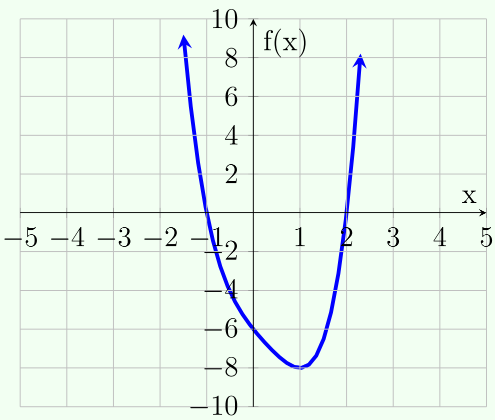
x	0	1	2	3	4	5	6	7	8
$p(x)$	21	18	15	12	9	6	3	0	-3

Solution:

Based on the values that we are given in the table, the outputs $p(x)$ are consistently decreasing. Since each consecutive value of $p(x)$ is lower than the preceding one (e.g. $p(0) > p(1)$), the function is not increasing. So this function is decreasing on its entire domain.

Example 2d: Graph

Given the following graph for the function $f(x)$, find the intervals where the function is increasing or decreasing.



Solution:

Again, we examine the output values as we move along our function from left to right. Notice that the function values are going down until we get to $x = 1$, after which the outputs start going up. This information tells us that $f(x)$ is increasing on the interval $(1, \infty)$ and decreasing on the interval $(-\infty, 1)$.

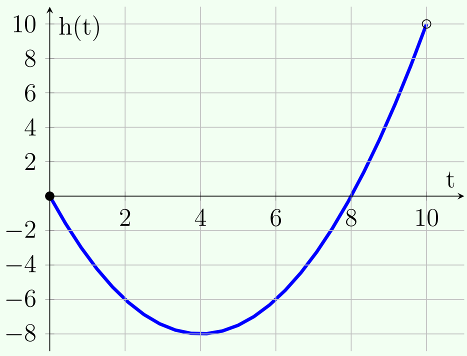
As we've seen, some functions are increasing or decreasing, and some are both. Sometimes however we have a function that is neither increasing or decreasing. For instance, the constant function $y = 2$ is neither increasing nor decreasing.

Putting It All Together

Let's look at one final example that combines all of these parts together and asks us to interpret what our answers mean in this case.

Example 3

The graph below represents the temperature in Celsius of an object, $h(t)$, t minutes after the start of an experiment.



(a) Find the domain and range for this function.

(b) Find the horizontal and vertical intercepts.

(c) Find the interval(s) where the function is positive and the interval(s) where the function is negative.

(d) Find $h(2)$. Interpret the meaning of this value in the context of the problem.

(e) Is the temperature of the object increasing, decreasing, neither, or both? How can you tell?

Solution:

(a) The domain for this function is $[0, 10)$ and the range is $[-8, 10)$.

(b) The horizontal intercepts for this function are $x = 0, 8$, i.e. $(0, 0)$ and $(8, 0)$. The vertical intercept is $y = 0$, i.e. $(0, 0)$.

(c) Since our domain is $[0, 10)$ and the horizontal intercepts are at $x = 0, 8$, we can see that the function is negative on $(0, 8)$ and positive for x in $(8, 10)$.

(d) Here, $h(2) = -6$. This means that 2 minutes after the start of the experiment, the temperature of the object was -6° Celsius.

(e) Both. The temperature of the object decreases for the first 4 minutes, so the function is decreasing on the interval $(0, 4)$. Then the temperature starts to increase, so the function is increasing on $(4, 10)$.